variation in internal pressure; these may be readily calculated from the present analysis. Further, it is quite possible to extend this type of approach to study more complicated balloon shapes if the need arises.

References

[1] Daskin, W. and Feldman, L., "The characteristics of two-dimensional sails in hypersonic flow," J. Aeronaut. Sci. **25**, 53–55 (1958).

[2] Boyd, E. A., "Aerodynamic characteristics of a hypersonic parachute," College of Aeronautics, Rept. 152 (1961).

[3] Rossettos, J. N., "Nonlinear membrane solutions for symmetrically loaded deep membranes of revolution," NASA Langley Research Center, TN D-3297 (1966).

[4] Sanders, J. L., "Nonlinear theories for thin shells," Quart. Appl. Math. **21**, 21–36 (1963).

[5] McShera, J. T., "Aerodynamic drag and stability characteristics of towed inflatable decelerators at supersonic speeds," NASA Langley Research Center, TN D-1601 (1963).

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Systematic Determination of Simplified Gain Scheduling Programs

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Given the control-loop gain settings for best handling qualities at each individual flight condition of interest, this paper presents a direct procedure for determining which simplified gain programs would yield the best performance for their level of complexity, and the approximate numerical values for these simplified gain schedules. Since the approach is suitable for digital mechanization, this phase of control system design can be largely automated.

Nomenclature

K,x,y,w	= variables
Z	= the numerical value of a function
i,j,g	= running indexes
I,J,G	= upper limits on indexes (i), (j), and (g), respectively
T_{ij}	= summation of (K) values over all indexes but (i) and (j); (T) would imply the summation over all (K) values with no exceptions, etc.
M_i	= measure of the importance of the ith function
SS	= summation of (K^2) values over all indexes
KP	= roll rate feedback to aileron
KR	= yaw rate feedback to rudder
KCF	= stick crossfeed of roll command into rudder

Introduction

GIVEN a set of flight condition variables (such as Mach number, angle of attack, dynamic pressure, center of gravity location, etc.), control-loop gain settings yielding the best handling qualities at each individual flight condition can be directly selected. However, the manner in which the gain scheduling should be simplified to achieve a practical level of airborne mechanization complexity with a minimum degradation in handling qualities is not so obvious. The technique presented in this article offers a direct method of doing this. The first section to follow presents a standard form for a functional relationship with two independent variables. Continuing with the two-variable case, the second section derives measures of the importance of each functional component to the dependent variable value, and develops expressions for computing these measures. The third section extends the two-variable case results to three variables, making the extension to four or more variables evident. Next, the logic for selecting control configurations from the computed data and an example application are presented. Finally, possible selection logic refinements and the manner in which numerical gain approximations are computed are discussed, in addition to summarizing the approach.

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Standard Functional Form

Given a dependent variable (K), which is a function of two independent variables (x) and (y), the functional relationship can always be expressed in two alternate forms as follows:

$$K = f(x,y) \qquad \text{or} \qquad K = \langle K\rangle + f(x) + f(y) + f(x,y)$$

For any set of (x) and (y) values, the values of the functions on the right-hand side of the latter expression can always be evaluated. For example, define the symbols (Z), (Z_i), (Z_j), and (Z_{ij}) as follows: Z = the value of the const $\langle K\rangle$; Z_i = the value of $f(x)$ for the ith value of (x); Z_j = the value of $f(y)$ for the jth value of (y); Z_{ij} = the value of $f(x,y)$ for the ith value of (x) and the jth value of (y). Then the latter expression becomes

$$K_{ij} = Z + Z_i + Z_j + Z_{ij}$$

By definition, the summations of the functions (Z_i), (Z_j), and (Z_{ij}) over their respective indexes are to be zero, i.e.,

$$\sum_i Z_i \equiv \sum_j Z_j \equiv \sum_i Z_{ij} \equiv \sum_j Z_{ij} \equiv 0$$

Symbolizing the maximum value of the (i) index as (I), and the maximum value of the (j) index as (J), the (Z_i) values can be derived as follows:

$$\sum_i K_{ij} = \sum_i Z + \sum_i Z_i + \sum_i Z_j + \sum_i Z_{ij} = IZ + IZ_j$$

$$Z_j = \left(\sum_i K_{ij} \Big/ I\right) - Z$$

where

$$Z = \left(\sum_i \sum_j K_{ij}\right) \Big/ IJ\Big)$$

Similarly, it follows that

$$Z_i = \left(\sum_j K_{ij} \Big/ J\right) - Z \qquad Z_{ij} = K_{ij} - Z - Z_i - Z_j$$

The practical significance of this standard functional form can best be illustrated by an example. Let the variable (K_{ij}) represent that value of a control-loop gain which yields the best handling qualities for a vehicle at a particular Mach number (x_i) and angle of attack (y_i). Thus, the (K_{ij}) values represent the ideal values for the gain over the Mach number and angle-of-attack spectrum to be flown. However, to obtain these values would require scheduling the gain as a two-dimensional function of Mach number and angle of attack. The second form of the relation expresses these ideal gain values as the sum of simpler functions, i.e., $\langle K \rangle \triangleq Z$: a const; $f(x) \triangleq Z_i$: a function of Mach number only; $f(y) \triangleq Z_j$: a function of angle of attack only; and $f(x,y) \triangleq Z_{ij}$: a two-dimensional function of Mach number and angle of attack. Therefore, this form displays more readily the importance to the gain values (and thus indirectly to the handling qualities) of each of the independent variables and functional relationships. The next section derives expressions that quantitatively measure the relative importance of each component.

Relative Importance of Components

A convenient measure of the relative importance of each component can be determined as follows. Squaring the (K_{ij}) values and summing over all (i) and (j) gives

$$\begin{aligned} K_{ij}^2 &= (Z + Z_i + Z_j + Z_{ij})^2 \\ &= Z^2 + 2ZZ_i + 2ZZ_j + 2ZZ_{ij} + Z_i^2 + 2Z_iZ_j + \\ &\qquad 2Z_iZ_{ij} + Z_j^2 + 2Z_jZ_{ij} + Z_{ij}^2 \end{aligned}$$

$$\sum_i \sum_j K_{ij}^2 = IJZ^2 + \sum_i \sum_j Z_i^2 + \sum_i \sum_j Z_j^2 + \sum_i \sum_j Z_{ij}^2$$

Dividing through by the left-hand side gives

$$1.0 = \left(IJZ^2 \Big/ \sum_i \sum_j K_{ij}^2\right) + \left(\sum_i \sum_j Z_i^2 \Big/ \sum_i \sum_j K_{ij}^2\right) + \left(\sum_i \sum_j Z_j^2 \Big/ \sum_i \sum_j K_{ij}^2\right) + \left(\sum_i \sum_j Z_{ij}^2 \Big/ \sum_i \sum_j K_{ij}^2\right)$$

The terms on the right-hand side represent the relative contribution of each component to the total sum-of-squares of the gain values, and, as such, are a reasonable measure of the relative importance of each component in obtaining the gains yielding the best handling qualities. Computationally simpler expressions for these terms will now be derived. Defining the following terms:

$$T_{ij} \triangleq K_{ij} \qquad T_i \triangleq \sum_j K_{ij} \qquad T_j \triangleq \sum_i K_{ij}$$

$$T \triangleq \sum_i \sum_j K_{ij} \qquad SS \triangleq \sum_i \sum_j K_{ij}^2$$

and since

$$Z = \left(\sum_i \sum_j K_{ij} \Big/ IJ\right) \triangleq \frac{T}{IJ}$$

the previously derived expression for (Z_i) can be written as follows:

$$Z_i = T_i/J - T/IJ$$

$$Z_i^2 = \frac{T_i^2}{J^2} - 2\frac{T_i}{J}\frac{T}{IJ} + \frac{T^2}{(IJ)^2}$$

$$\begin{aligned} \sum_i \sum_j Z_i^2 &= \sum_i \sum_j \frac{T_i^2}{J^2} - 2\frac{T}{IJ^2} J \sum_i T_i + \frac{T^2}{IJ} \\ &= \sum_i \sum_j \frac{T_i^2}{J^2} - 2\frac{T^2}{IJ} + \frac{T^2}{IJ} \\ &= \sum_i \frac{T_i^2}{J} - \frac{T^2}{IJ} \end{aligned}$$

Therefore, the relative contribution of the (Z_i) component can be computed from the expression

$$M_i \triangleq \left(\sum_i \sum_j Z_i^2 \Big/ \sum_i \sum_j K_{ij}^2\right) = \left(\sum_i \frac{T_i^2}{J} - \frac{T^2}{IJ}\right) \Big/ SS$$

By similar reasoning, the expressions for the other relative contributions can be shown to be as follows:

$$M \triangleq \left(IJZ^2 \Big/ \sum_i \sum_j K_{ij}^2\right) = \frac{T^2/IJ}{SS}$$

$$M_j \triangleq \left(\sum_i \sum_j Z_j^2 \Big/ \sum_i \sum_j K_{ij}^2\right) = \left(\sum_j \frac{T_j^2}{I} - \frac{T^2}{IJ}\right) \Big/ SS$$

$$M_{ij} \triangleq \left(\sum_i \sum_j Z_{ij}^2 \Big/ \sum_i \sum_j K_{ij}^2\right) = 1.0 - M - M_i - M_j$$

For the case of a control configuration containing several gains, the contribution of each component to each gain would be computed as described. Then, the average contribution of each component to all the gains would be calculated to obtain a measure of the importance of an individual component to the performance obtained with the set of gains.

Generalization to Three Variables

For the case of three independent variables we can write

$$K = f(x,y,w)$$

or

$$K = Z + Z_i + Z_j + Z_g + Z_{ij} + Z_{ig} + Z_{jg} + Z_{ijg}$$

The measures of the relative importance of the components are given by the following equation:

$$\begin{aligned} 1.0 = &\left(IJGZ^2 \Big/ \sum_i \sum_j \sum_g K_{ijg}^2\right) + JG\left(\sum_i Z_i^2 \Big/ \sum_i \sum_j \sum_g K_{ijg}^2\right) + IG\left(\sum_j Z_j^2 \Big/ \sum_i \sum_j \sum_g K_{ijg}^2\right) + \\ &IJ\left(\sum_g Z_g^2 \Big/ \sum_i \sum_j \sum_g K_{ijg}^2\right) + G\left(\sum_i \sum_j Z_{ij}^2 \Big/ \sum_i \sum_j \sum_g K_{ijg}^2\right) + J\left(\sum_i \sum_g Z_{ig}^2 \Big/ \sum_i \sum_j \sum_g K_{ijg}^2\right) + \\ &I\left(\sum_i \sum_g Z_{ig}^2 \Big/ \sum_i \sum_j \sum_g K_{ijg}^2\right) + \\ &\left(\sum_i \sum_j \sum_g Z_{ijg}^2 \Big/ \sum_i \sum_j \sum_g K_{ijg}^2\right) \end{aligned}$$

The measures would be computed from the following expressions:

$$M \triangleq IJG\frac{Z^2}{SS} = \frac{T^2/IJG}{SS}$$

$$M_i \triangleq JG\left(\sum_i Z_i^2 \Big/ SS\right) = \left(\sum_i \frac{T_i^2}{JG} - \frac{T^2}{IJG}\right) \Big/ SS$$

$$M_j \triangleq IG\left(\sum_j Z_j^2 \Big/ SS\right) = \left(\sum_j \frac{T_j^2}{IG} - \frac{T^2}{IJG}\right) \Big/ SS$$

$$M_g \triangleq IJ\left(\sum_g Z_g^2 \Big/ SS\right) = \left(\sum_g \frac{T_g^2}{IJ} - \frac{T^2}{IJG}\right) \Big/ SS$$

Table 1 Categories of complexity for the 3-variable case

	Category 1	2	3	4	5	6
No. of sensed variables	1	2	3	2	3	3
Type of functions mechanized (dimensions)	1	1	1	1 & 2	1 & 2	1, 2, & 3
No. of possible configurations	3	3	1	3	1	1

$$M_{ij} \triangleq G\left(\sum_i\sum_j Z_{ij}{}^2\Big/SS\right) = \left[\left(\sum_i\sum_j \frac{T_{ij}{}^2}{G} - \frac{T^2}{IJG}\right)\Big/SS\right] - M_i - M_j$$

$$M_{ig} \triangleq J\left(\sum_i\sum_g Z_{ig}{}^2\Big/SS\right) = \left[\left(\sum_i\sum_g \frac{T_{ig}{}^2}{J} - \frac{T^2}{IJG}\right)\Big/SS\right] - M_i - M_g$$

$$M_{jg} \triangleq I\left(\sum_j\sum_g Z_{jg}{}^2\Big/SS\right) = \left[\left(\sum_j\sum_g \frac{T_{jg}{}^2}{I} - \frac{T^2}{IJG}\right)\Big/SS\right] - M_j - M_g$$

$$M_{ijg} \triangleq \left(\sum_i\sum_j\sum_g Z_{ijg}{}^2\Big/SS\right) = 1.0 - M - M_i - M_j - M_g - M_{ij} - M_{ig} - M_{jg}$$

The extension to the cases of four or more variables follows directly. For those familiar with Analysis of Variance techniques, the standard functional form and derived expressions are, of course, those of a factorial test design (with no replication) for two and three crossed factors, respectively.

Configuration Selection Logic

This section describes a procedure for using the computed measures of relative importance to determine the simplified configurations yielding the best performance for their level of complexity. The specific numerical values used for making decisions should be considered only as representative of values which could be used. The procedure is as follows:

1) Eliminate all one- and two-dimensional functions whose contribution is less than (0.05) to their associated gains and any three-dimensional function that does not contribute (0.15).

Table 2 Desirable values of three SAS gains at twenty-eight flight conditions

i	j	KP for $g = 1$	KP for $g = 2$	KR for $g = 1$	KR for $g = 2$	KCF for $g = 1$	KCF for $g = 2$
1	1	1.44	0.34	0.88	0.22	0.17	0.05
1	2	0.69	1.97	4.05	0.17	0.59	0.00
2	1	3.31	1.07	0.80	0.36	0.07	0.05
2	2	0.71	0.36	2.63	0.84	0.07	0.31
3	1	3.56	1.35	0.80	0.33	0.11	0.09
3	2	0.82	0.40	3.09	1.03	0.24	0.38
4	1	4.14	1.43	0.85	0.39	0.11	0.09
4	2	0.83	0.33	2.20	0.76	0.39	0.54
5	1	2.30	0.69	1.57	0.89	−0.03	−0.05
5	2	0.54	0.25	3.95	1.37	0.43	0.90
6	1	1.80	0.53	1.95	1.11	−0.19	−0.21
6	2	0.78	0.42	5.34	1.71	0.11	0.50
7	1	1.99	0.54	1.76	1.04	−0.20	−0.20
7	2	0.95	0.55	6.50	1.99	0.08	0.39

Table 3 Relative importance measures of the functional components making up each gain value

	Z	Z_i	Z_j	Z_g	Z_{ij}	Z_{ig}	Z_{jg}	Z_{ijg}
KP	0.59	0.03	0.11	0.09	0.07	0.02	0.07	0.01
KR	0.56	0.08	0.12	0.14	0.02	0.02	0.06	0.01
KCF	0.30	0.11	0.33	0.02	0.09	0.08	0.02	0.06
Av.	0.48	0.07	0.19	0.08	0.06	0.04	0.05	0.03

2) Of the two-dimensional functions remaining, eliminate any that contribute less than (0.10), unless the two corresponding one-dimensional functions remain after step 1.

3) Compute the performance indexes (average percentage of sum of squares of gain values accounted for) of the candidate configurations. Eliminate those configurations whose indexes are more than (0.05) less than the highest index within their complexity category.

4) Eliminate any of the remaining configurations with performance indexes of less than (0.50).

The objective of the first two steps of the procedure is to eliminate from further consideration those functional components that do not increase performance sufficiently to justify the mechanization complexity they would add. Conversely, then, the functions remaining after these steps are worthwhile mechanizing if the sensed variables are available. Because of this, the result of the first two steps is to reduce the candidate configurations to, at most, 12 (for 3 variables). These 12 candidates fall into one of 6 categories of complexity (for the 3-variable case) (Table 1). The objective of the third step is to eliminate any definitely inferior candidates (of the twelve) within each category, leaving only those configurations that yield the best performance for their level of complexity. The fourth step eliminates any configurations whose absolute level of performance is almost certain to be inadequate.

Implicit in this selection procedure is the assumption that the handling qualities obtainable from a simplified configuration are directly related to the percentage of the sum-of-squares of the ideal gain values obtained with the configuration. Although there is necessarily a reasonable degree of correlation, the correspondence is not one-to-one. It is because of this lack of precision that performance indexes within 5 percentage points of each other are considered equal. The following section presents an example of this procedure.

Example Application

For a practical example, the lateral mode stability augmentation system gains for a lifting body over a 28-condition flight spectrum were analyzed. The three SAS gains were KP, KR, and KCF, defined as follows: KP: roll rate feedback to aileron; KR: yaw rate feedback to rudder; and KCF: stick crossfeed of roll command into rudder. The desirable values of these gains at each of the 28 flight conditions were determined and appear in Table 2. The (i) subscript has been used to designate each of seven Mach numbers, the (j) index, each of 2 angles of attack, and the (g) index, each of 2 values of dynamic pressure. The measures of the relative importance of each functional component were computed and are presented in Table 3. Table 4 presents the results of applying steps one and two of the

Table 4 Results of applying first and second selection steps to Table 3 data

	Z	Z_i	Z_j	Z_g	Z_{ij}	Z_{ig}	Z_{jg}	Z_{ijg}
KP	0.59	...	0.11	0.09	...	...	0.07	...
KR	0.56	0.08	0.12	0.14	...	...	0.06	...
KCF	0.30	0.11	0.33	...	0.09	...	...	...
Av	0.48	0.06	0.19	0.08	0.03	0.00	0.04	0.00

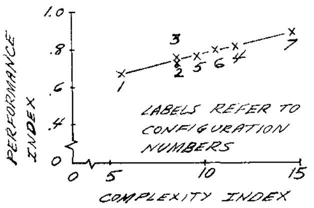

Fig. 1 Performance vs complexity.

selection logic to the data in Table 3. Table 5 presents the results of applying step 3 to the data in Table 4. Step 4 of the logic does not effect the results in this example. Therefore, Table 5 represents the simplified configurations yielding the best performance in their complexity category. Two additional refinements could be made to the selection logic to further screen the possibilities. These are discussed in the following section.

Selection Logic Refinements

The first and simplest refinement would be to use a somewhat more precise measure of system complexity than complexity category, to better evaluate the performance index return relative to system complexity. Such a yardstick could be constructed as follows. Given a specific vehicle and set of control loops, the complexity of the resulting control system will vary with the number (and type) of flight condition variables to be sensed, and the number and type of mathematical relations to be mechanized. A rough quantitative measure of the relative complexity of possible mechanizations can be established by considering both the hardware design problems and operational reliability associated with sensing the possible variables and making the possible airborne computations. For example, a designer may feel that sensing Mach number accurately is five times more difficult than mechanizing a constant gain amplifier for airborne use. In addition, he may feel that sensing Mach number accurately is about as difficult as sensing dynamic pressure accurately. On the basis of this type of reasoning, a rough, quantitative complexity scale could be constructed as shown in Table 6. Using this scale of values, the complexity of each possible system mechanization can be assigned a numerical value as follows:

$$C_i = \Sigma \text{ (scale values for the variables to be sensed) } + \Sigma \text{ (scale values for the functions to be mechanized)}$$

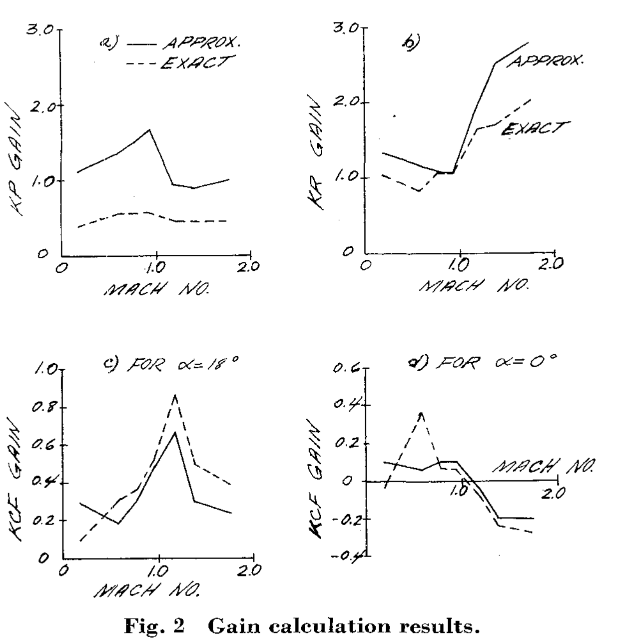

Fig. 2 Gain calculation results.

Table 5 Results of applying selection logic to example

Configuration	Performance index	Gain	Functional relationship
		KP	$Z + Z_j$
1	0.67	KR	$Z + Z_j$
		KCF	$Z + Z_j$
		KP	$Z + Z_j$
2	0.73	KR	$Z + Z_i + Z_j$
		KCF	$Z + Z_i + Z_j$
		KP	$Z + Z_j + Z_g$
3	0.75	KR	$Z + Z_j + Z_g$
		KCF	$Z + Z_j$
		KP	$Z + Z_j + Z_g$
4	0.81	KR	$Z + Z_i + Z_j + Z_g$
		KCF	$Z + Z_i + Z_j$
5	0.76	KP	$Z + Z_j$
		KR	$Z + Z_i + Z_j$
		KCF	$Z + Z_i + Z_j + Z_{ij}$
6	0.79	KP	$Z + Z_j + Z_g + Z_{jg}$
		KR	$Z + Z_j + Z_g + Z_{jg}$
		KCF	$Z + Z_j +$
7	0.88	KP	$Z + Z_j + Z_g + Z_{jg}$
		KR	$Z + Z_i + Z_j + Z_g + Z_{jg}$
		KCF	$Z + Z_i + Z_j + Z_{ij}$

This index satisfies the intuitive requirement of increasing as the number of variables to be sensed increases, the number of computations to be made increases, and the complexity of computations increases. Using this yardstick, the complexity indexes of the final configurations of Table 5 were computed and are presented in Table 7. In these computations, the sum of Z_i, Z_j, and Z_{ij} functions (for example) is considered as a single 2-*d* function (Z_{ij}), since presumably it would be mechanized in this fashion. A plot of performance vs complexity index for these configurations is presented in Fig. 1. The figure indicates that performance is linearly related to complexity for this example. Thus, there is no configuration yielding a particularly high level of performance for its level of complexity. The choice of configuration will therefore depend primarily upon determining the simplest system yielding satisfactory handling qualities. The level of complexity required depends entirely upon the degree to which handling quality criteria were satisfied by the ideal gain settings. For example, if the best that could be done was marginal, very little degradation is allowable, and the minimum acceptable performance index in this case might be 95%. On the other hand, if the ideal performance were more than satisfactory, a performance index of perhaps 80% might be acceptable. A quantitative estimate of the minimum performance index that will yield satisfactory handling qualities is the second refinement, which could be made in the selection procedure. The basis for this estimate would be to decrease all gain values by perhaps 15% from their ideal values and determine if the resulting handling qualities over the complete flight spectrum were still satisfactory. If so, a configuration with a performance index of (0.75) or better would probably yield satisfactory results. If the results were marginal, a performance index of (0.80) or more probably would be required.

Table 6 Quantitative complexity scale

Item	Scale value
Accurate Mach number sensing	1.0
Accurate dynamic pressure sensing	1.0
Accurate angle-of-attack sensing	2.0
Const gain amplifier	0.2
One-dimensional function generation	1.0
Two-dimensional function generation	3.0
Three-dimensional function generation	5.0

Table 7 Complexity indices of selected configurations

Configuration	Performance index	Complexity index
1	0.67	5.6
2	0.73	8.6
3	0.75	8.6
4	0.81	11.6
5	0.76	9.6
6	0.79	10.6
7	0.88	14.6

Gain Profile Calculation

Once the most promising simplified configurations have been determined from estimates of their handling quality capabilities, it remains to determine the actual handling qualities obtainable from them. This latter step requires that the numerical values of the specified functional components yielding the best handling qualities be determined. For a particular configuration (defined in Table 8), the values of the functional components which produced the best fit of the vehicle roots (in a least-squares sense) to the initial root values obtained at the individual flight conditions were computed and are presented in Fig. 2 as the curves labeled "exact." A good approximation to these profiles can be obtained from the expressions derived for Z_i, Z_j, etc. These approximate profiles have been computed for the configuration defined in Table 8 and are presented in Fig. 2 as the curves labeled "approximate." The figure shows that the approximation is within about 10% for important terms, and degrades to 50% for minor terms.

Table 8 Gain profile example configuration

Performance index: 0.69
Complexity index: 8.6
$KP = Z + Z_i$
$KR = Z + Z_i$
$KCF = Z + Z_i + Z_j + Z_{ij}$

Summary and Conclusions

In summary, then, a method has been presented which 1) starts with the gain settings for best handling qualities at individual flight conditions for a given vehicle and set of control loops; 2) from these settings develops quantitative estimates of the performance obtainable with simpler gain programming; 3) uses these estimates to systematically pick out the most promising simplified configurations; and 4) using the original setting data, develops expressions to compute the approximate numerical values of the gain programs (for the promising configurations) which will produce the best handling qualities.

Although the required computations can be easily accomplished by hand in one day, they can be readily programmed on a digital computer. The input to the computer would be the individual flight condition gain settings and the output would be the most promising simplified configurations, along with the approximate numerical values for their gain programs.